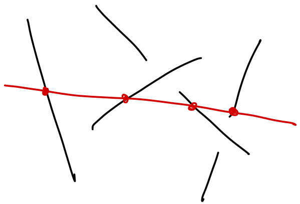
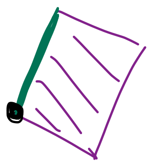
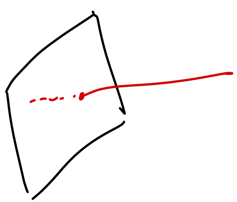
Flag varieties & Schubert Calculus

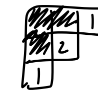
Part II

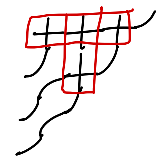


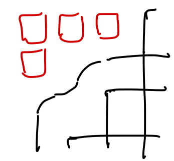


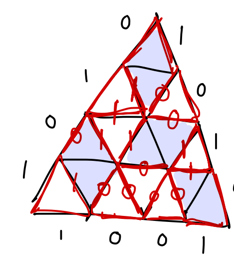


$H^*(G/B)$









$\mathfrak{G}_w$

Jake Levinson

Simon Fraser University

Combinatorics Days in Covilhã

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Flags

A complete flag $\mathcal{F}_\bullet$ in $\mathbb{C}^n$ is a sequence of subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n, \quad \dim F_i = i.$$

$$Fl_n = \{\text{complete flags in } \mathbb{C}^n\}.$$ complete flag variety.

We can represent $\mathcal{F}_\bullet$ as the top-justified row spans of an $n \times n$ invertible matrix:

ex:

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{array}{l} \} F_1 \\ \} F_2 \\ \} F_3 = \mathbb{C}^3 \end{array}$$

Note: We can do row ops downwards without changing $\mathcal{F}_\bullet$.

That is,

$$GL_n \xrightarrow[\text{row spans}]{\text{top-justified}} Fl_n = B \backslash GL_n$$

Borel subgroup of lower-triangular matrices (row ops ↓)

✱ General setting:

G = Lie group

B = Borel subgp (maximal solvable subgp)

G/B = generalized flag variety

P = parabolic subgp $\quad G/P$

$$Gr(k,n) = GL_n \Big/ \left[\begin{array}{c|c} I & * \\ \hline & I \end{array}\right]$$

Flag Schubert cells

Analog of echelon form: 0's below and left of some permutation matrix.

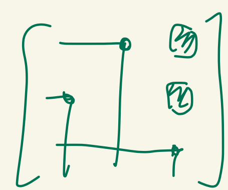
e.g. $w = 3124 \in S_4$

$$\begin{bmatrix} 0 & 0 & 1 & * \\ 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1's in $(i, w(i))$.

0's below, left.

e.g. $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1/2 \\ 1 & 0 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$ $w = 213$



Rothe diagram of w

(start from top row)

e.g. A generic matrix will reduce to

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad w = 123 \text{ (identity)}.$$

<u>Fact</u>: Every flag has a <u>unique</u> representative in this "permutation form".

$$\rightsquigarrow \Omega_w^\circ = \left\{ \text{Flags rep'd by matrices } \begin{array}{l} \text{with } 1\text{'s in } (i, w(i)) \ \forall i \\ 0\text{'s below, left of } 1\text{s} \\ *\text{'s elsewhere} \end{array} \right\}$$

is a <u>(flag) Schubert cell</u>.

$\Omega_w := \overline{\Omega_w^\circ}$ is a (flag) <u>Schubert variety</u>.

Facts:

① $Fl_n = \bigsqcup_{w \in S_n} \Omega_w^\circ$

② $\Omega_w^\circ \cong \mathbb{C}^{\binom{n}{2} - \ell(w)}$, $\ell(w)$ = length of w as a permutation.

That is, $\operatorname{codim}(\Omega_w) = \ell(w)$.

③ $\Omega_w = \bigsqcup_{w \le v} \Omega_v^\circ$ where $w \le v$ in right Bruhat order.

covering relation: $v = w \cdot (ij)$ and $\ell(v) = \ell(w) + 1$

no other 1's in here

$$\begin{array}{c} i \\ \vdots \\ j \end{array} \underbrace{\begin{bmatrix} 1 & & * \\ & & \\ & & 1 \end{bmatrix}}_{w} \le \begin{array}{c} i \\ \vdots \\ j \end{array} \underbrace{\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix}}_{v}.$$

Ex: Schubert stratification of Fl_3:

w_0 long element

$$\begin{tikzcd}[column sep=small]
 & 321 \arrow[dl, dash] \arrow[dr, dash] & \\
231 \arrow[d, dash] \arrow[drr, dash] & & 312 \arrow[d, dash] \arrow[dll, dash] \\
213 \arrow[dr, dash] & & 132 \arrow[dl, dash] \\
 & w=123 &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{bmatrix} & 1 & * \\ & & 1 \\ 1 & & \end{bmatrix} \arrow[d, dash] \arrow[drr, dash] & & \begin{bmatrix} & & 1 \\ 1 & * & \\ & 1 & \end{bmatrix} \arrow[d, dash] \arrow[dll, dash] \\
\begin{bmatrix} & 1 & * \\ 1 & & * \\ & & 1 \end{bmatrix} \arrow[dr, dash] & & \begin{bmatrix} 1 & * & * \\ & & 1 \\ & 1 & \end{bmatrix} \arrow[dl, dash] \\
 & \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} &
\end{tikzcd}$$

(largest cell)

Right Bruhat order on S_3.
(strong)

Cohomology of Fl_n

Fact: $H^*(Fl_n) \cong \bigoplus_{w \in S_n} \mathbb{Z} \cdot [\Omega_w]$. $\qquad H^{2k} = \bigoplus_{w : \ell(w) = k} \mathbb{Z}[\Omega_w]$.

That is, the Schubert classes form an <u>additive basis</u> for H^*.

But, the structure constants $[\Omega_w] \cdot [\Omega_v] = \sum_{\ell(u) = \ell(w) + \ell(v)} c_{wv}^{u} [\Omega_u]$

are NOT KNOWN in general! (But are $\in \mathbb{Z}_{\geq 0}$.)

Schubert polynomials

Polynomial model for $H^*(Fl_n)$ (Borel):

Let $e_k(x_1, \ldots, x_n) = k^{th}$ elementary symmetric polynomial

$$= \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$ squarefree monomials

e.g. $e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$

Thm (Borel 1953): $H^*(Fl_n) \cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{(e_1, e_2, \ldots, e_n)}$

(For geometers:

$x_i \leftrightarrow -c_1(F_i/F_{i-1})$

Chern class of a line bundle of Fl_n.)

Nice Fact (easy induction): Monomial basis $\{x_1^{k_1} \cdots x_n^{k_n} : 0 \le k_i \le n-i \text{ for all } i\}$.

$\hookrightarrow k_n = 0$.

e.g. Fl_3:

$$\begin{matrix} x_1^2 & x_1^2 x_2 \\ x_1 & x_1 x_2 \\ 1 & x_2 \end{matrix}$$

Nicer Fact: Compatible with $Fl_n \xhookrightarrow{i} Fl_{n+1}$

$$(F_1 \subset \cdots \subset F_n = \mathbb{C}^n) \overset{i}{\longmapsto} (F_1 \subset \cdots \subset \underset{\mathbb{C}^n}{F_n} \subset \underset{\mathbb{C}^{n+1}}{F_{n+1}})$$

$$\begin{tikzcd}
x_i \arrow[d, maps to] & \mathbb{Z}[x_1, \dots, x_n, x_{n+1}] \arrow[r, two heads] \arrow[d, "x_{n+1} = 0"'] & H^*(Fl_{n+1}) \arrow[d, "i^*"] & {[\Omega_w]} \arrow[d, maps to, "i^*"] \\
x_i & \mathbb{Z}[x_1, \dots, x_n] \arrow[r, two heads] & H^*(Fl_n) & {[\Omega_w]}
\end{tikzcd}$$

$w \in S_n \subseteq S_{n+1}$

How can we represent a Schubert class $[\Omega_w]$ in the monomial basis?

Lemma: Using $H^*(\mathcal{F}\ell_{n+1}) \xrightarrow{i^*} H^*(\mathcal{F}\ell_n)$, $i^*[\Omega_w] = [\Omega_w]$ $\quad w \in S_n \subseteq S_{n+1}$

Implies: $\exists!$ polynomial $\underset{(S)}{\mathfrak{S}_w} \in \mathbb{Z}[x_1, \ldots, x_n]$ $\quad (w \in S_n)$

mapping to $[\Omega_w] \in H^*(\mathcal{F}\ell_m)$ for all $m \geq n$.

Q: What is $\mathfrak{S}_w$?

Def: The ith <u>divided difference operator</u> is

$$\partial_i : \mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n],$$

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}.$$

$s_i = (i\ i{+}1) \in S_n$

$s_i f = f(x_1, \ldots, x_{i-1}, \overbrace{x_{i+1}, x_i}, x_{i+2}, \ldots, x_n)$

numerator is antisymmetric in x_i, x_{i+1}, hence divisible by $x_i - x_{i+1}$.

These satisfy

1. $\partial_i \partial_j = \partial_j \partial_i \quad |i-j|>1$
2. $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$
3. $\partial_i^2 = 0$.

$S_n = \langle s_i : \cdot\ s_i s_j = s_j s_i \ \ldots,\ \cdot$ braid rel, $\cdot\ s_i^2 = id \rangle$

Def: For $w \in S_n$, a reduced word is a minimal-length factorization

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \quad \ell = \ell(w), \qquad s_i = (i\ i{+}1) \in S_n.$$

By ①-③, the composition $\partial_w := \partial_{i_1} \circ \cdots \circ \partial_{i_\ell}$ doesn't depend on the choice of reduced word.

(non-reduced word: $\partial_{i_1} \circ \cdots \circ \partial_{i_\ell} = 0$)

Reminder on multiplying permutations:

e.g. $w = 461253$

✓ right-multiplying by s_i switches positions $i, i{+}1$ e.g. $w \cdot s_4 = 461523$

✗ left-multiplying by s_i switches values $i, i{+}1$ e.g. $s_4 w = 561243$

Def: Let $S_\infty := \bigcup_{n\geq 1} S_n$.

The Schubert polynomials are defined recursively by:

- (Base cases) For $w_0^{(n)} = (n\ n-1 \cdots 1)$, $\mathfrak{S}_{w_0^{(n)}} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1}$
- (Inductive cases) $\partial_i(\mathfrak{S}_w) =: \mathfrak{S}_{ws_i}$ if $\ell(ws_i) < \ell(w)$.

Note: $w_0^{(n)} s_1 s_2 \cdots s_n = w_0^{(n-1)}$, so $\mathfrak{S}_w$ is well-defined by the braid relations.

- Effectively $\mathfrak{S}_w = \partial_{w_0 w}(\mathfrak{S}_{w_0})$

  ↳ Start with $w_0 = n\ n-1 \cdots 1$, repeatedly switch elements to get $w = w_1 \cdots w_n$.

Example (S_4) Calculate $\mathfrak{S}_{1432}$.

e.g. $\partial_1 (x_1^5 x_2^2) = x_1^4 x_2^2 + x_1^3 x_2^3 + x_1^2 x_2^4$

$$
\begin{array}{c}
4321 \\
\downarrow s_3 \\
4312 \\
\downarrow s_2 \\
4132 \\
\downarrow s_1 \\
1432
\end{array}
$$

$$\mathfrak{S}_{4321} = x_1^3 x_2^2 x_3$$

$$\Big\downarrow \partial_3$$

$$x_1^3 x_2^2$$

$$\Big\downarrow \partial_2$$

$$x_1^3 (x_2 + x_3) = x_1^3 x_2 + x_1^3 x_3$$

$$\Big\downarrow \partial_1$$

$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

Example: S_3

$$\begin{tikzcd}[column sep=small]
& w_0 = 321 \arrow[dl, dash, "s_1"'] \arrow[dr, dash, "s_2"] & \\
231 \arrow[d, dash, "s_2"'] & & 312 \arrow[d, dash, "s_1"] \\
213 \arrow[dr, dash, "s_1"'] & & 132 \arrow[dl, dash, "s_2"] \\
& 123 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
& \mathfrak{S}_{w_0} = x_1^2 x_2 \arrow[dl, dash, "\partial_1"'] \arrow[dr, dash, "\partial_2"] & \\
\mathfrak{S}_{231} = x_1 x_2 \arrow[d, dash, "\partial_2"'] & & \mathfrak{S}_{312} = x_1^2 \arrow[d, dash, "\partial_1"] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, dash, "\partial_1"'] & & \mathfrak{S}_{132} = x_1 + x_2 \arrow[dl, dash, "\partial_2"] \\
& \mathfrak{S}_{id} = 1 &
\end{tikzcd}$$

Caution: This shows fewer edges than right Bruhat order.

This graph is called right weak order.

Thm (Billey-Jockusch-Stanley '93):

Positive formula for coefficients of $\mathfrak{S}_w$.

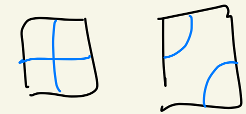
Note. each $\mathfrak{S}_w$ has a distinct leading term in revlex order.

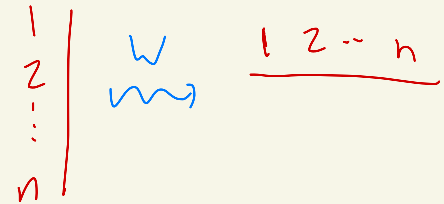
$$\left\{ \mathfrak{S}_w : w \in \bigcup_{n \geq 1} S_n \right\} \text{ is a basis for } \mathbb{Z}[x_1, x_2, x_3, \ldots].$$

Pictorial approach:

"Pipe dreams" (Fomin-Kirillov '97, Bergeron-Billey '93)

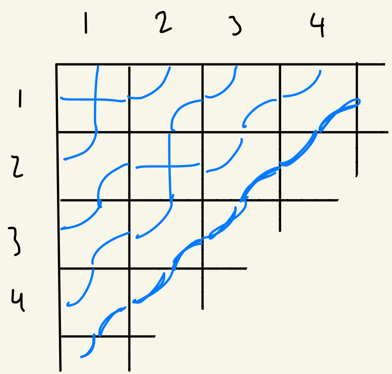
Def: *pipe dream* for $w \in S_n$:

- Tiles: 

- Connect 

- Each pair of pipes may only cross ≤ 1 time. ($\Leftrightarrow$ reduced words)

- **Weight**: $wt(P) = \prod_{\boxplus \text{ tiles}} x_{row(\boxplus)}$

$P =$ 

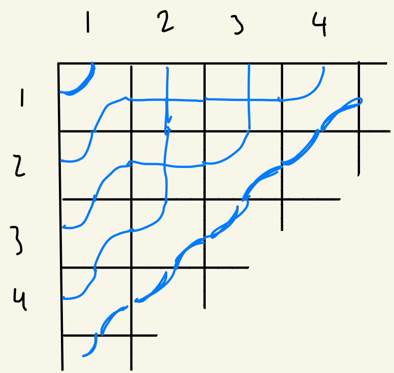
$w = 2143 \in S_4$

$wt(P) = x_1 x_2$

(BJS, BB, FK)

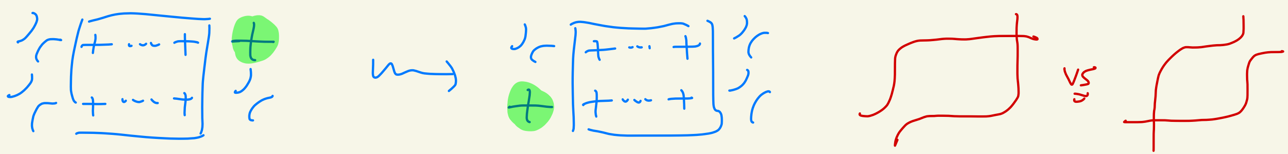
Theorem: $\mathfrak{S}_w = \sum_{P \text{ for } w} wt(P)$.

Ex: $w = 1432$. Greedy top-justified pipe dream:

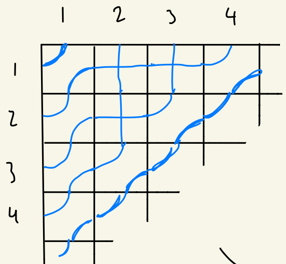


Weight $= x_1^2 x_2$.

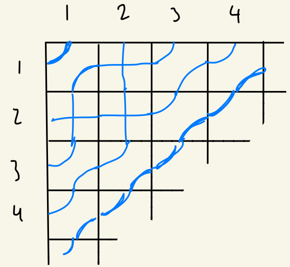
Thm: Generate all others using "chute moves" from this one:



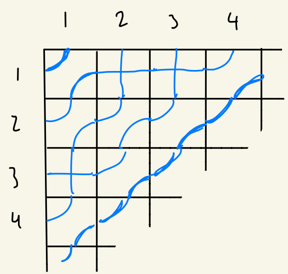
Ex (cont'd)



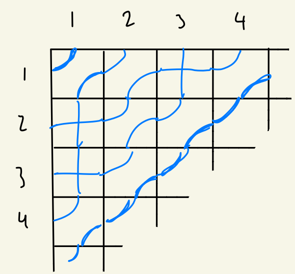
$x_1^2 x_2$



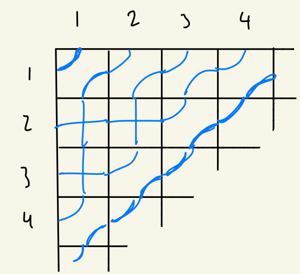
$x_1 x_2^2$



$x_1^2 x_3$



$x_1 x_2 x_3$



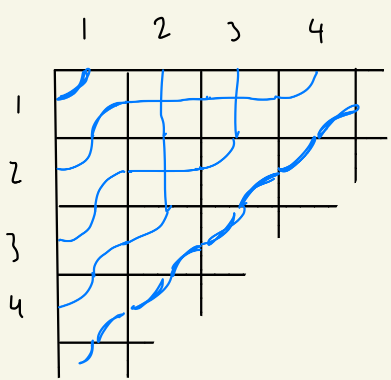
$x_2^2 x_3$

$$\mathfrak{G}_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

Double Schubert polynomial:

$$\mathfrak{S}_w(x_1, x_2, \dots ; \underbrace{y_1, y_2, \dots}_{\text{"equivariant variables"}}) = \sum_{\text{pipe dreams}} \prod_{\boxplus \text{ tiles}} \left(x_{\mathrm{row}(\boxplus)} - y_{\mathrm{col}(\boxplus)} \right)$$

Corresponds to a class in equivariant cohomology $H_T^*(Fl_n)$

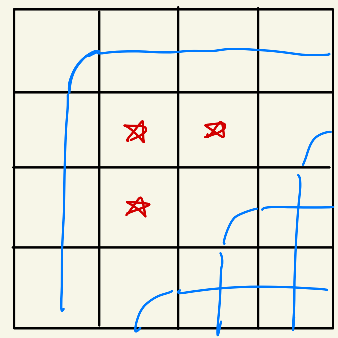


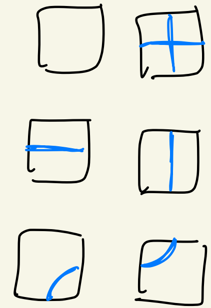
$\rightsquigarrow (x_1 - y_2)(x_1 - y_3)(x_2 - y_2)$

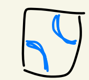
More modern formula: Bumpless pipe dreams (Lam-Lee-Shimozono '18)

$w = 1432$ Rothe diagram

(left-right reversed from before)



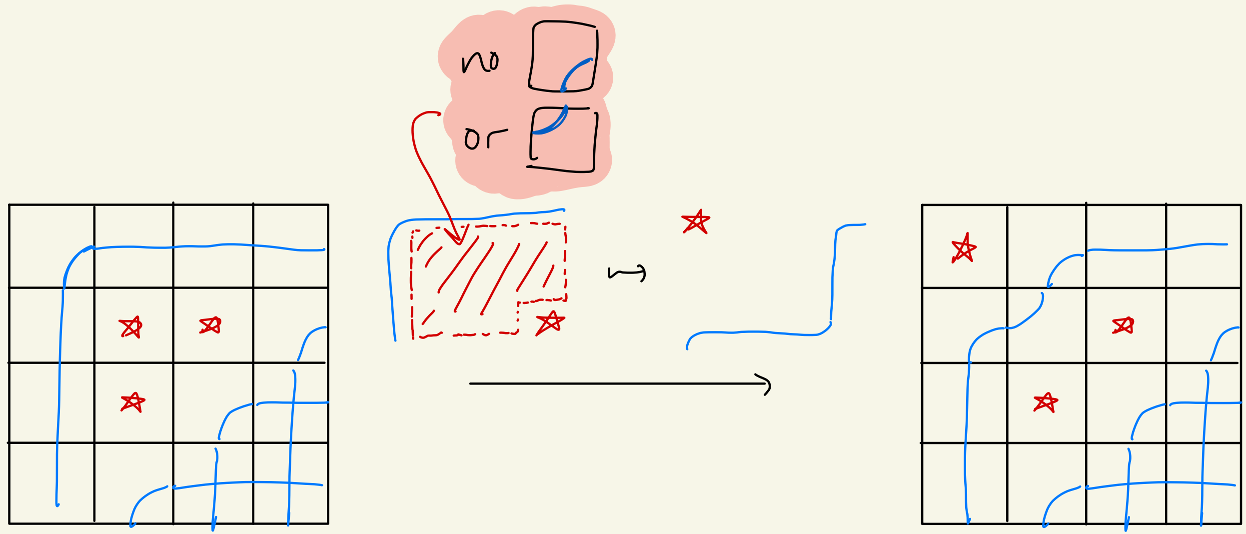
Tiles: 

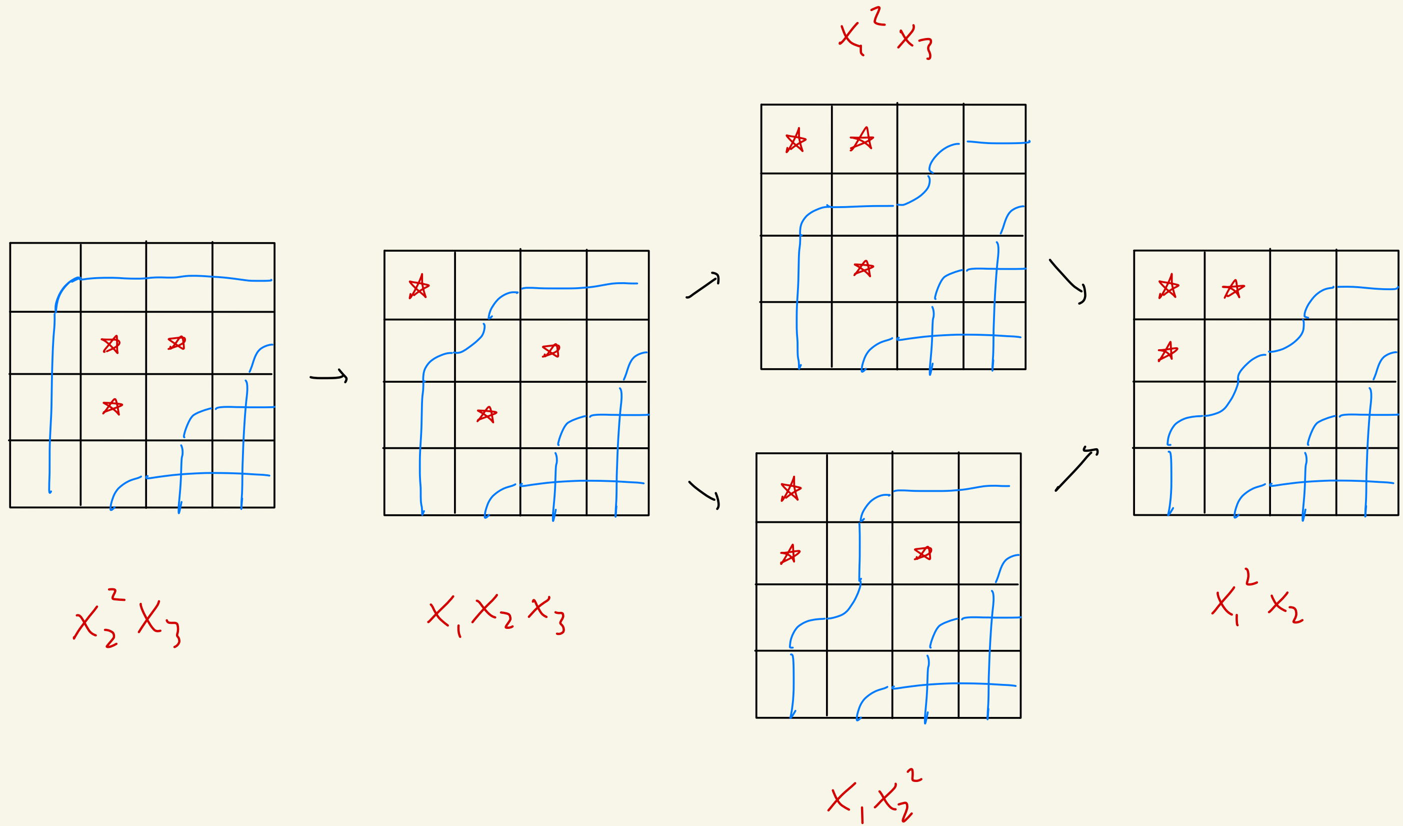
(no bumps  !)

$$\text{weight} = \prod_{\boxed{\star}\ \text{box}} x_{\text{row}(\boxed{\star})} \qquad x_2^2 x_3$$

($x_{\text{row}(\boxed{\star})} - y_{\text{col}(\boxed{\star})}$ for double Schubert polynomials)

Generate all other BPDs using "droop moves":




$x_1^2 x_3$
$x_2^2 x_3$
$x_1 x_2 x_3$
$x_1^2 x_2$
$x_1 x_2^2$

Significance of pipe dreams in algebra/geometry:

Def: For $w \in S_n$, $M_w^\circ = \left\{ M = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} : \begin{array}{l} M \text{ represents an element} \\ \text{of } X_w \subseteq Fl_n \end{array} \right\} \subseteq \mathbb{C}^{n^2}$.

$M_w = \overline{M_w^\circ}$ is a matrix Schubert variety.

$$\begin{array}{ccc} M_w^\circ & \xhookrightarrow{\text{closure}} & M_w \subseteq \mathbb{C}^{n^2}. \\ \Big\downarrow {\scriptstyle \text{row spans}} & & \\ X_w \subseteq Fl_n & & \end{array}$$

Thm (Fulton): M_w is defined by "obvious" determinants (read off from the Rothe diagram).

Thm (Knutson-Miller '05) Pipe dreams list the irreducible components of the Groebner degeneration of M_w under an antidiagonal monomial order.

Thm (Klein-Weigandt '22) Bumpless pipe dreams: same, for diagonal term orders, & counted with multiplicity.

Structure constants

For Schubert polynomials,

$$\mathfrak{S}_w \mathfrak{S}_v = \sum_u c_{wv}^u \mathfrak{S}_u \quad \text{for } \underline{\text{some}} \ c_{wv}^u \in \mathbb{Z}_{\geq 0}.$$

<u>No</u> combinatorial rule for c_{wv}^u is known in general.

"Flag Littlewood-Richardson rule".

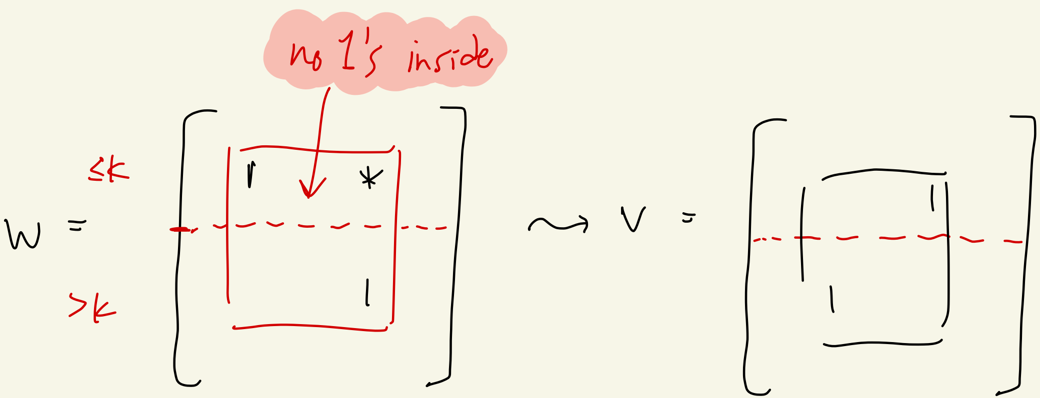
Monk's Rule

Analog of the Pieri rule $\left(s_\lambda \cdot s_{\square\square\square\square} = \sum s_{\lambda'} \cdots\right)$

$$\mathfrak{S}_w \cdot \underbrace{\mathfrak{S}_{s_k}}_{=\,(x_1+\cdots+x_k)} = \sum_v \mathfrak{S}_v \quad \text{where} \quad \left.\begin{array}{l} v = w\cdot(i\,j) \\ \ell(v) = \ell(w)+1 \end{array}\right\} \begin{array}{l}\text{covering relation}\\ \text{in Bruhat order}\end{array}$$

and $i \le k < j$

Pictorially:



Ex: Calculate $\mathfrak{S}_{3142} \cdot \mathfrak{S}_{s_2}$. $k=2$

$$\begin{bmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{bmatrix} = \mathfrak{S}_{4132} + \mathfrak{S}_{3412} + \mathfrak{S}_{3241}.$$

Other cases where c_{wv}^{u} is known

- w, v are k-Grassmannian (only descent is $w_k > w_{k+1}$) $\quad \mathfrak{S}_w = s_\lambda$
- 2-step partial flag varieties (Buch, Kresch, Purbhoo, Tamvakis '14)
  ↳ 2 descents in same positions
- 3-step (Knutson, Zinn-Justin '17, '21)
- "separated descents" (Huang '23)
  "almost separated" (K, Z-J '23)
- "inverse Grassmannian" $\mathfrak{S}_{u^{-1}} \mathfrak{S}_{v^{-1}}$ u, v Grassmannian (Pechenik-Weigandt '22)

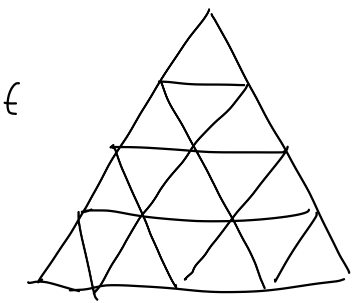
  coeffs $\in \{1, 0\}$.

Epilogue: Variations on the subject

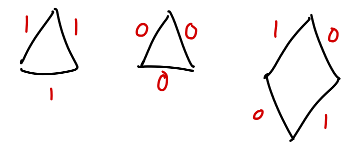
(2001)

Knutson-Tao-Woodward puzzles: Alternate LR rule for $Gr(k,n)$.

A puzzle is a tiling of

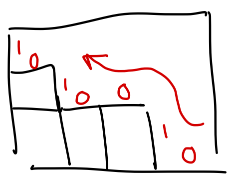


by



and rotations, but not reflections.

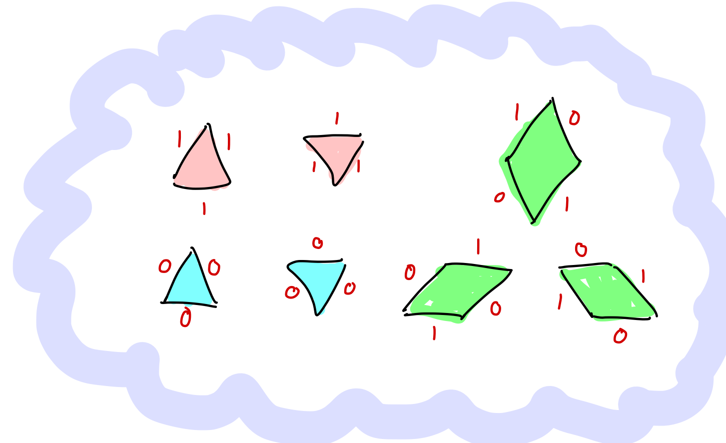
Write $\lambda =$



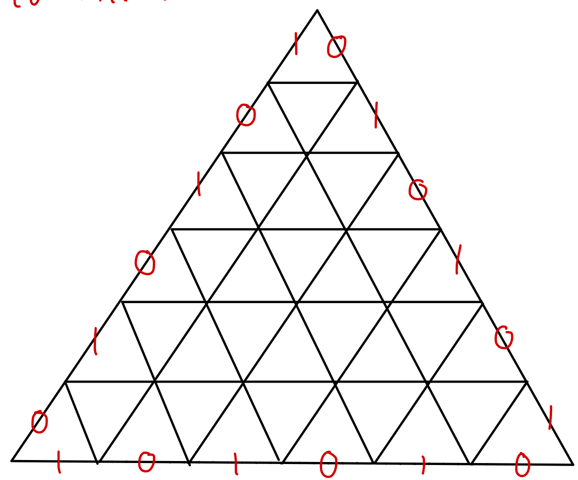
$\longleftrightarrow$ 0100101

Theorem: $c_{\lambda\mu}^{\vee}$ = # puzzles with

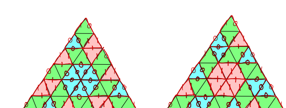
Ex: $C^{\ydiagram{1,2,3}}_{\ydiagram{1,2},\ \ydiagram{1,2}} = C^{101010}_{010101,\ 010101} = 2.$



Try to fill in!



Ans:

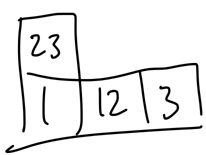


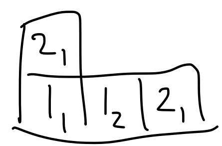
Other kinds of cohomology

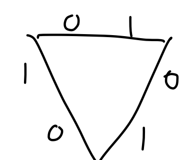
K-theory $K(X)$: for Euler characteristic instead of counting

- Basis $[\mathcal{O}_\lambda]$, multiplication $[\mathcal{O}_\lambda]\cdot[\mathcal{O}_\mu] = (\text{LR rule}) + (\text{higher-order terms})$

- Structure constants for $K(Gr(k,n))$:
  set-valued tableaux (Buch), genomic tableaux (Pechenik-Yong), puzzles + "K-theory piece"







(can't be rotated).

Equivariant cohomology: $H_T^*(X)$

based on the action of the torus $(\mathbb{C}^*)^n \circlearrowright Gr(k,n)$, Fl_n

Quantum cohomology $QH^*(X)$

for counting the number of curves passing through given subvarieties

Other spaces:

- Other Lie types: Orthogonal & Lagrangian Grassmannians, G/P.

  shifted tableaux, signed permutations.